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ON SETS OF HYPERREAL NUMBERS

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ABSTRACT

We construct a field $^*R$ of hyperreal numbers so that the field R of real numbers is embedded as a subfield of $^*R$. The set $^*R$ can be regarded as a metric space containing the set R as a discrete subset and we can generalize this property.

Key words and phrases. Ultrafilter, Hyperreal number, Metric space.

1. **Introduction and preliminaries.** In the 1960s, Abraham Robinson showed that the set R of real numbers can be regarded as a subject of a set $^*R$ of hyperreal numbers which contains infinitely small and large numbers. We can contend the set $^*R$ is a metric space containing the set R as a discrete subset. In the present paper we would like to generalize this property.

We shall first state some fundamental properties on filters. According to Comfort and negrepontis [1], we shall give the following definition.

Definition 1.1. Let I_i be an infinite set and $\mathcal{F}_i$ be a filter on I_i for each $i = 1,2,3$. We define sets $\mathcal{F}_1 \cdot \mathcal{F}_2$, $(\mathcal{F}_1 \cdot \mathcal{F}_2) \cdot \mathcal{F}_3$, $\mathcal{F}_1 \cdot \mathcal{F}_2 \cdot \mathcal{F}_3$, and $\mathcal{F}_1 \cdot (\mathcal{F}_2 \cdot \mathcal{F}_3)$ as follows.

$$\mathcal{F}_1 \cdot \mathcal{F}_2 = \{A \in P(I_1 \times I_2) \mid \{y_1 \in I_1 \mid \{y_2 \in I_2 \mid (y_1, y_2) \in A\} \in \mathcal{F}_2\} \in \mathcal{F}_1\}$$

$$(\mathcal{F}_1 \cdot \mathcal{F}_2) \cdot \mathcal{F}_3 = \{A \in P(I_1 \times I_2 \times I_3) \mid \{(y_1, y_2) \in I_1 \times I_2 \mid \{y_3 \in I_3; (y_1, y_2, y_3) \in A\} \in \mathcal{F}_3\} \in \mathcal{F}_1 \cdot \mathcal{F}_2\},$$

$$\mathcal{F}_1 \cdot \mathcal{F}_2 \cdot \mathcal{F}_3 = \{A \in P(I_1 \times I_2 \times I_3) \mid \{y_1 \in I_1 \mid \{y_2 \in I_2 \mid \{y_3 \in I_3 \mid (y_1, y_2, y_3) \in A\} \in \mathcal{F}_3\} \in \mathcal{F}_2\} \in \mathcal{F}_1\},$$

$$\mathcal{F}_1 \cdot (\mathcal{F}_2 \cdot \mathcal{F}_3) = \{A \in P(I_1 \times I_2 \times I_3) \mid \{y_1 \in I_1 \mid \{(y_2, y_3) \in I_2 \times I_3 \mid (y_1, y_2, y_3) \in A\} \in \mathcal{F}_2 \cdot \mathcal{F}_3\} \in \mathcal{F}_1\}.$$

We immediately have the following proposition.

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Proposition 1.2.

(1.1) $\mathcal{F}_1 \cdot \mathcal{F}_2$ is a filter on $I_1 \times I_2$.

(1.2) $(\mathcal{F}_1 \cdot \mathcal{F}_2) \cdot \mathcal{F}_3$, $\mathcal{F}_1 \cdot \mathcal{F}_2 \cdot \mathcal{F}_3$ and $\mathcal{F}_1 \cdot (\mathcal{F}_2 \cdot \mathcal{F}_3)$ are filters on $I_1 \times I_2 \times I_3$ and we have

$$(\mathcal{F}_1 \cdot \mathcal{F}_2) \cdot \mathcal{F}_3 = \mathcal{F}_1 \cdot \mathcal{F}_2 \cdot \mathcal{F}_3 = \mathcal{F}_1 \cdot (\mathcal{F}_2 \cdot \mathcal{F}_3).$$

The following proposition was given in [3].

Proposition 1.3. Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be ultrafilters on I_1 and I_2 respectively. Then $\mathcal{F}_1 \cdot \mathcal{F}_2$ is an ultrafilter on $I_1 \times I_2$.

Proposition 1.4. Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be ultrafilters on I_1 and I_2 respectively, and let $\mathcal{F}_1$ or $\mathcal{F}_2$ be a free (ω – incomplete) ultrafilter, then $\mathcal{F}_1 \cdot \mathcal{F}_2$ is a free ultrafilter on $I_1 \times I_2$.

In order to prove Proposition 1.4, we use a lemma.

Lemma 1.5. Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be filters on I_1 and I_2 respectively. Then

(1.3) let $X_1 \in \mathcal{F}_1$ and $X_2 \in \mathcal{F}_2$, then $X_1 \times X_2 \in \mathcal{F}_1 \cdot \mathcal{F}_2$,

and

(1.4) let $X_1 \subset I_1$ and $X_2 \subset I_2$, and let $X_1 \notin \mathcal{F}_1$ or $X_2 \notin \mathcal{F}_2$, then $X_1 \times X_2 \notin \mathcal{F}_1 \cdot \mathcal{F}_2$.

Proof of Proposition 1.4. Let $\mathcal{F}_1$ be a free ultrafilter, then there exists A_n such that $A_n \in \mathcal{F}_1$ for each $n \in N$ and $\bigcap_{n=1}^{\infty} A_n \notin \mathcal{F}_1$, where N is the set of all positive integers.

Using Lemma 1.5, we have $A_n \times I_2 \in \mathcal{F}_1 . \mathcal{F}_2$ for each $n \in N$.

Since

$$\bigcap_{n=1}^{\infty} (A_n \times I_2) = (\bigcap_{n=1}^{\infty} A_n) \times I_2 \text{ and } \bigcap_{n=1}^{\infty} A_n \notin \mathcal{F}_1,$$

we have

$$(\bigcap_{n=1}^{\infty} A_n) \times I_n \notin \mathcal{F}_1 \cdot \mathcal{F}_2.$$

Therefore $\mathcal{F}_1 \cdot \mathcal{F}_2$ is a free ultrafilter. Q.E.D.

Let K be a nonempty set and

$$a\,(y_1, \ldots, y_n) \ , \ b\,(y_1, \ldots, y_n) \in \prod_{(y_1, \ldots, y_n) \in I_1 \times \ldots \times I_n} K .$$

We define a relation $R\,(n)$, by

$$a\,(y_1, \ldots, y_n)\, R\,(n)\, b\,(y_1, \ldots, y_n),$$

if and only if

$$\{(y_1, \ldots, y_n) \in I_1 \times \ldots \times I_n ;\, a\,(y_1, \ldots, y_n) = b\,(y_1, \ldots, y_n)\} \in \mathcal{F}_1 . \mathcal{F}_2 \ldots \mathcal{F}_n.$$

The relation $R(n)$ is an equivalence relation.

We would like to use a notation

$$\prod_{(y_1, \ldots, y_n) \in I_1 \times \ldots \times I_n} K / \mathcal{F}_1 . \mathcal{F}_2 \ldots \mathcal{F}_n$$

in order to express the quotient space

$$\prod_{(y_1, \ldots, y_n) \in I_1 \times \ldots \times I_n} K \;/\; R(n)$$

The quotient class determined by a function $a\,(y_1, \ldots, y_n)$ will be denoted by $[a\,(y_1, \ldots, y_n)]$.

Theorem 1.6. The following formula is valid.

$$\prod_{y_1 \in I_1} \Big(\prod_{y_2 \in I_2} K / \mathcal{F}_2 \Big) / \mathcal{F}_1 = \prod_{(y_1, y_2) \in I_1 \times I_2} K \;/\; \mathcal{F}_1 \cdot \mathcal{F}_2.$$

Proof. If

$$(\tilde{a}\,(y_2))\,(y_1) \in \prod_{y_1 \in I_1} \Big(\prod_{y_2 \in I_2} K \Big),$$

then we can contend

$$(\tilde{a}\,(y_2))\,(y_2) \in \prod_{(y_1, y_2) \in I_1 \times I_2} K,$$

and we write

$$(\tilde{a}\,(y_2))\,(y_1) = a\,(y_1, y_2).$$

We immediately have this result by the following fact. Let

$$[[\tilde{a}\,(y_2)]\,(y_1)],\ [[\tilde{b}\,(y_2)]\,(y_1)] \in \prod_{y_1 \in I_1} \Big(\prod_{y_2 \in I_2} K / \mathcal{F}_2 \Big) / \mathcal{F}_1,$$

then we have

$$[[\tilde{a}\,(y_2)]\,(y_1)] = [[\tilde{b}\,(y_2)]\,(y_1)]$$

$$\Leftrightarrow \{y_1 \in I_1;\ [\tilde{a}(y_2)]\,(y_1) = [\tilde{b}\,(y_2)]\,(y_1)\} \in \mathcal{F}_1$$

$$\Leftrightarrow \{y_1 \in I_1;\ \{y_2 \in I_2;\ (\tilde{a}\,(y_2))\,(y_1) = (\tilde{b}(y_2))\,(y_1)\} \in \mathcal{F}_2\} \in \mathcal{F}_1$$

$$\Leftrightarrow \{(y_1, y_2) \in I_1 \times I_2;\ a\,(y_1, y_2) = b\,(y_1, y_2)\} \in \mathcal{F}_1 \cdot \mathcal{F}_2. \qquad \text{Q.E.D.}$$

Corollary 1.7. The following formula is valid.

$$\prod_{y_1 \in I_1} \Big(\prod_{y_2 \in I_2} \Big(\ldots \Big(\prod_{y_n \in I_n} K / \mathcal{F}_n \Big) \ldots \Big) / \mathcal{F}_2 \Big) / \mathcal{F}_1 = \prod_{(y_1, \ldots, y_n) \in I_1 \times \ldots \times I_n} K \;/\; \mathcal{F}_1 \ldots \mathcal{F}_n.$$

2. On sets of hyperreal numbers

Now we shall give the following definition.

Definition 2.1. Let $R^+ = \{y \in R \mid y > 0\}$ and let $\mathrm{F} = \{(0, y) \mid y \in R^+\}$. Then F has the finite intersection property. We shall denote by $\mathcal{F}$ one of the ultrafilters containing F. The filter $\mathcal{F}$ is a free ultrafilter.

We shall use the following notations:

$$\mathcal{F}^1 = \mathcal{F} \text{ and } \mathcal{F} \cdot \mathcal{F}^n = \mathcal{F}^{n+1} \text{ for } n \in N,$$

$$(R^+)^1 = R^+ \text{ and } R^+ \times (R^+)^n = (R^+)^{n+1} \text{ for } n \in N,$$

$$^{0*}R = R,\ ^{1*}R = {}^*R = \prod_{y_1 \in R^+} R/\mathcal{F}, \text{ and}$$

$$^{n*}R = \prod_{(y_1, \ldots, y_n) \in (R^+)^n} R / \mathcal{F}_1 \ldots \mathcal{F}_n \qquad \text{for } n \in N.$$

We immediately have the following theorem.

Theorem 2.2. $^{n*}R = {}^*(^{(n-1)*}R)$ for $n \in N$.

An element of the set $^{n*}R$ is called a hyperreal number. The set $^{n*}R$ is made into a commutative ordered field by defining the addition, the subtraction, the product, the quotient and the order in the usual way.

We define absolute value in $^{*n}R$ as follows.

Definition 2.3. If $x = [x\ (y_1, \ldots, y_n)] \in {}^{*n}R$, then $|\,x\,| = [\,|\,x\ (y_1, \ldots, y_n)\,|\,]$.

We have the following theorem immediately.

Theorem 2.4. $^{n*}R$ is a metric space with a usual metric

$$d\ (x, y) = |\ y - x\,| \text{ for } x, y \in {}^{n*}R.$$

Definition 2.5 (Infinitesimal). We shall say that $\varepsilon = [\varepsilon\ (y_1, \ldots, y_n)] \in {}^{n*}R$ is infinitesimal or infinitesimal small if for every $\delta \in \mathrm{R}^+$ we have

$$\{(y_1, \ldots, y_n) \in (R^+)^n |\ |\varepsilon(y_1, \ldots, y_n)| < \delta\} \in \mathcal{F}^n \qquad (2.1)$$

When ε and δ satisfy condition (2.1), we write $|\ \varepsilon\ | < \delta$.

Proposition 2.6. Let $\delta \in R^+$ and $\varepsilon = [\varepsilon\ (y_1, \ldots, y_n)] \in {}^{*n}R$, then we have

$$\{(y_1, \ldots, y_n) \in (R^+)^n |\ |\varepsilon(y_1, \ldots, y_n)| < \delta\} \in \mathcal{F}^n$$

$$\Leftrightarrow \{y_1 \in R^+ |\ \{(y_2, \ldots, y_n) \in (R^+)^{n-1} \ |\ |\varepsilon\ (y_1, \ldots, y_n)| < \delta\} \in \mathcal{F}^{n-1}\} \in \mathcal{F}$$

$$\Leftrightarrow \{y_1 \in R^+ |\ \{y_2 \in R^+ |\ \{(y_3, \ldots, y_n) \in (R^+)^{n-2};\ |\ \varepsilon\ (y_1, \ldots, y_n)| < \delta\} \in \mathcal{F}^{n-2}\} \in \mathcal{F}\} \in \mathcal{F}\}$$

$$\Leftrightarrow \ldots$$

$$\Leftrightarrow \{y_1 \in R^+ |\ \{y_2 \in R^+ |\ \ldots \{y_n \in R^+ |\ |\varepsilon(y_1, \ldots, y_n)\ | < \delta\} \in \mathcal{F} \ldots\} \in \mathcal{F}\} \in \mathcal{F}$$

$$\Leftrightarrow \{(y_1, y_2) \in (R^+)^2 |\ \{(y_3, \ldots, y_n) \in (R^+)^{n-2} |\ |\varepsilon\ (y_1, \ldots, y_n)| < \delta\} \in \mathcal{F}^{n-2}\} \in \mathcal{F}^2$$

$$\Leftrightarrow \ldots$$

$$\Leftrightarrow \{(y_1, \ldots, y_{n-1}) \in (R^+)^{n-1} |\ \{y_n \in R^+ |\ |\ \varepsilon\ (y_1, \ldots, y_n)\ | < \delta \in \mathcal{F}\} \in \mathcal{F}^{n-1}.$$

The proof is easy, and we omit it.

Proposition 2.7. Let $\varepsilon = [\varepsilon(y_n)] \in {}^*R$ and $\delta \in R^+$ which satisfy a condition

$$\{y_n \in R^+ \mid |\varepsilon(y_n)| < \delta\} \in \mathcal{F}.$$

Then we have $$\{(y_1, ..., y_n) \in (R^+)^n \mid |\varepsilon(y_n)| < \delta\} \in \mathcal{F}^n.$$

Proof. Let $A = \{y_n \in R^+ \mid |\varepsilon(y_n)| < \delta\}$, then $A \in \mathcal{F}$.

Since $$\{(y_1, ..., y_n) \in (R^+)^n \mid |\varepsilon(y_n)| < \delta\} = (R^+)^{n-1} \times A,$$

$$(R^+)^{n-1} \in \mathcal{F}^{n-1} \text{ and } A \in \mathcal{F},$$

we have $$\{(y_1, ..., y_n) \in (R^+)^n \mid |\varepsilon(y_n)| < \delta\} \in \mathcal{F}^{n-1} \cdot \mathcal{F} = \mathcal{F}^n.$$ Q.E.D.

Let a, x, $\varepsilon \in {}^{n*}R$ and $\varepsilon > 0$. We define a set $U(a, \varepsilon)$ by

$$U(a, \varepsilon) = \{x \in {}^{n*}R \mid |x - a| < \varepsilon\}.$$

Proposition 2.8. Let $a = [a(y_1, ..., y_{n-1})]$, $x = [x(y_1 ..., y_{n-1})] \in {}^{(n-1)*}R$ and let $\varepsilon = |\varepsilon(y_n)|$ be a positive infinitesimal, then

$$U(a, \varepsilon) = \{a\}.$$

Proof. Let $a \neq x$, and let

$$B = \{(y_1, ..., y_{n-1}) \in (R^+)^{n-1} \mid |x(y_1, ..., y_{n-1}) - a(y_1, ..., y_{n-1})| > 0\}$$

then we have $B \in \mathcal{F}^{n+1}$.

Since

$$\{(y_1, ..., y_{n-1}) \in (R^+)^{n-1} \mid \{y_n \in R^+ \mid |x(y_1, ... y_{n-1}) - a(y_1, ..., y_{n-1})| > \varepsilon(y_n)\} \in \mathcal{F}\} \supset B,$$

we have
$\{(y_1, ..., y_n) \in (R^+)^n \mid |x(y_1, ..., y_{n+1}) - a(y_1, ..., y_{n-1})| > \varepsilon(y_n)\} \in \mathcal{F}^n$, which shows $|x - a| > \varepsilon$.

Clearly we have $U(a, \varepsilon) \ni a$. Hence we have

$$U(a, \varepsilon) = \{a\}.$$ Q.E.D.

Using Proposition 2.8 we have the following theorem.

Theorem 2.9. A metric space ${}^{(n-1)*}R$ is a discrete subspace of a metric space ${}^{n*}R$, for every $n \in N$.

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